Pdf-02 (01) H-1049 Dr. Satish Kr Gupta
Algebra —spg22967@gmail.com

Q.I. Define set of right cosets of a normal subgroup of a group G.

Ans. Let H be a normal subgroup of a group G then $\frac{G}{H} = \{Ha \mid a \in G\}$ is called set of all right cosets of H in G.

Q2. Show

$\left\{ \frac{G}{H}, \text{ multiplication of right cosets of H in G} \right\}$

where H is a normal subgroup of a group G

Ans. Let H be a normal subgroup of a group G then ~~then~~ ~~Ha~~ $\frac{G}{H} = \{Ha \mid a \in G\}$ is a set of all right cosets of H in G.

To prove $\frac{G}{H}$ forms a group

Closure property. Let $Ha, Hb \in \frac{G}{H}$, $a, b \in G$

then $Ha\,Hb = Hab \in \frac{G}{H}$ as $(ab) \in G$, $[\because H \triangleleft G]$

$\Rightarrow \frac{G}{H}$ is closed, $\forall\ Ha, Hb \in \frac{G}{H}$.

Associativity. Let $Ha, Hb, Hc \in \frac{G}{H}$, $a, b, c \in G$

then to show $(HaHb)Hc = Ha(HbHc)$, $\forall\ Ha, Hb, Hc \in \frac{G}{H}$

$LHS = (HaHb)Hc$

$LHS = (Hab)Hc$ $\left[\because (HaHb) = Hab \text{ as } H \triangleleft G\right]$

$LHS = H(ab)c$

$LHS = Ha(bc)$ $[\because (ab)c = a(bc)]$

$LHS = Ha(Hbc)$

Scanned by CamScanner

$LHS = Ha(HbHc)$
$= RHS$
$\Rightarrow (HaHb)Hc = Ha(HbHc).$

Existance of identity element. Let $Ha \in \frac{G}{H}$, $a \in G$ then $\exists\, He \in \frac{G}{H}$ called identity element such that

$HaHe = Hae$.
$HaHe = Ha$ $[\because ae = a$, e is identity in $G]$

which shows that $He = H$ is an identity element in $\frac{G}{H}$ as $He = H$
because $h \in H \Rightarrow Hh = H$
put $h = e$

Existance of inverse element.
Let $He \in \frac{G}{H}$, $Ha \in \frac{G}{H}$ then $\exists\, Ha^{-1} \in \frac{G}{H}$ called inverse of $\frac{G}{H}$ such that

$HaHa^{-1} = Haa^{-1}$
$HaHa^{-1} = He$ $[\because aa^{-1} = e]$
$HaHa^{-1} = H$

$\Rightarrow Ha^{-1}$ is the inverse of Ha.
$\Rightarrow$ Inverse of each element of $\frac{G}{H}$ exists.

So, $\frac{G}{H}$ forms a group
$\Rightarrow \frac{G}{H}$ is a quotient group as it is a set of right cosets of H in G.

Scanned by CamScanner

Pdf-02, (3) Algebra, Dr Satish Kr Gupta

Example. If G is finite group and H is any normal
01 Subgroup of G then prove $O\left(\frac{G}{H}\right) = \frac{O(G)}{O(H)}$.

Ans. Let H be a normal subgroup of a finite group
then $\frac{G}{H} = \{Ha \mid a \in G\}$

Now, $o\left(\frac{G}{H}\right)$ = Number of distinct element of $\frac{G}{H}$.

$o\left(\frac{G}{H}\right)$ = index of H in G

$o\left(\frac{G}{H}\right) = \frac{O(G)}{O(H)}$, by Lagrange's theorem.

, proved.

Example 02. Show that every quotient group of an abelian group is abelian and the converse is not true.
of the group G.

Solution. Let $\frac{G}{H}$ be a ~~gro~~ quotient group

then $\frac{G}{H} = \{Ha \mid a \in G\}$

To prove $\frac{G}{H}$ is abelian

Let $Ha, Hb \in \frac{G}{H}$, $a, b \in G$

Now,

$(Ha)(Hb) = Hab$ $[\because H \Delta G]$

$(Ha)(Hb) = Hba$ $[\because ab = ba]$

$(Ha)(Hb) = (Hb)(Ha)$

$\Rightarrow \frac{G}{H}$ is abelian.

The converse is not true

i.e if $\frac{G}{H}$ is abelian then it is not necessary G is abelian

Let $G = P_3$, $H = A_3$

then $\frac{P_3}{A_3}$ is abelian while P_3 is not abelian

, P_3 = set of all permutations defined on a set having 3 elements which is a group.

A_3 = A subset of P_3 which is alternating group.

Example. (03) If N is a normal subgroup of a group G, $a \in G$ of order n then prove that the order m of Normalizer of a in $\frac{G}{N}$ is a divisor of order of a.

Solution. Let N be a normal subgroup of a group G.

Let $a \in G$ such that $a^n = e$, e is identity in G

then
$$\frac{G}{N} = \{ Na \mid a \in G \}$$

Let $o[Na] = m$

than to prove m is a divisor of n.

It is given $o(a) = n \Rightarrow a^n = e$ —— (1)

$\therefore Na^n = Ne$ $[\because a^n = e]$

$Na^n = N$ —— (2)

But $Na^n = N \underbrace{a.a.a. \ldots a}_{n \text{ times}}$

$= \underbrace{(Na)(Na)(Na) \ldots (Na)}_{n \text{ times}}$

$Na^n = (Na)^n$ —— (3)

From (2) and (3) implies

$(Na)^n = N$, the identity element of $\frac{G}{N}$

$\therefore O(Na) = n$

But it is given $O(Na) = m$, m is least

so m must divide n

$\Rightarrow O(Na)$ is a divisor of $o(a)$, $a \in G$.

Note. If $o(a) = n \Rightarrow a^n = e \Rightarrow (a^n)^k = e$

$\Rightarrow a^{nk} = e \Rightarrow n \mid nk$

ie (n) divides (multiple of n)

(4) Show that every quotient group of a cyclic group is cyclic and the converse is not true

Soln Let G be a cyclic group and a be a generator of G.

i.e. $G = \{a\}$, $a \in G$.

Let H be a normal subgroup of G.

then $\frac{G}{H}$ is a quotient group.

Let $a^n \in G$ then

$\frac{G}{H} = \{ Ha^n \mid a^n \in G$ for some integer $n\}$

To prove $\frac{G}{H}$ is cyclic

Let $Ha^n \in \frac{G}{H} \Rightarrow Ha^n = Ha.a.a. \dots a.$ (upto n terms)

$= (Ha)(Ha) \dots (Ha)$ (n terms)

$Ha^n = (Ha)^n.$

$\Rightarrow \frac{G}{H}$ is a cyclic group with generator as Ha.

Converse is true

ex $\frac{P_3}{A_3}$ is cyclic but P_3 is not a cyclic group.

(5) Let Z be the centre of a group G. let $a \in Z$ then prove the cyclic subgroup $\{a\}$ of G is normal in G.

Soln. let G be a group

Let Z be the centre of G.

then $Z = \{ z \in G \mid zx = xz, \forall x \in G\}$

Let $a \in Z$

and Let $H = \{a\}$ be a subgroup of G

To show $H \triangleleft G$

Let $x \in G$, $h \in H \Rightarrow h = a^n$ for some integer n

$\therefore xhx^{-1} = xa^n x^{-1} = (xax^{-1})^n$

$= (axx^{-1})^n$ $[\because a \in Z \Rightarrow ax = xa]$

$= (ae)^n = a^n \in H$

$\Rightarrow H \triangleleft G$, proved.

Scanned by CamScanner

(6) Let $a \in G$, G be a group. Show that the cyclic subgroup of G generated by a is normal subgroup of $N(a)$.

Solution. Let a be any element of a group G

Let H be a cyclic subgroup of G generated by a

i.e let $H = \{a\}$ be a cyclic subgroup of G

Now $N(a) = \{x \in G \mid ax = xa\}$

to prove $H \subseteq N(a)$

Let $h \in H \Rightarrow h = a^n$ for some integer n.

$ha = a^n \cdot a = a^{n+1} = a \cdot a^n = ha \Rightarrow h \in N(a)$

$\therefore h \in H \Rightarrow h \in N(a)$

$\Rightarrow H \subseteq N(a)$

Also $N(a)$ is a subgroup of G

$\Rightarrow H$ is a subgroup of $N(a)$

To show $H \triangleleft N(a)$

Let $x \in N(a)$, $h \in H$

$\Rightarrow xhx^{-1} = xa^nx^{-1} = (xax^{-1})^n$ $\quad [x \in N(a) \Rightarrow xa = ax]$

$= (axx^{-1})^n$

$= (ae)^n = a^n \in H$

$\Rightarrow H \triangleleft N(a)$, proved.

(7) Show two elements are conjugate iff they can be put in the form xy and yx respectively where x & y are suitable elements of G.

If Solution. Let G be a group. let $a, b \in G$

Suppose a & b are conjugate elements

To prove they can be put in the form of xy & yx respec.

$a \sim b \Rightarrow a = c^{-1}bc$ for some $c \in G$

Let $c^{-1}b = x$ & $c = y$ then $a = xy$ —— (1)

Again $yx = c(c^{-1}b) = \underbrace{(cc^{-1})b}_{=b}$

$\Rightarrow b = yx$ —— (2)

if part is proved

<u>only if part</u> Suppose $a = xy$ and $b = yx$

Now $b = yx \Rightarrow y^{-1}b = y^{-1}yx$

$\Rightarrow y^{-1}b = ex$

$\Rightarrow y^{-1}b = x$ —— (3)

~~$\Rightarrow y^{-1}by = xy$~~

Now, $\therefore a = xy$

$= (y^{-1}b)\,y$ [From (3)]

$a = y^{-1}by$

$\Rightarrow a \sim b \Rightarrow$ a and b are conjugate, proved.

(8) Give an example to show that in a group G the normalizer of an element is not necessarily a normal subgroup of G.

Solⁿ. Let $S = (a, b, c)$

then $P_3 = \{I, (a\,b), (b\,c), (c\,a), (a\,b\,c), (a\,c\,b)\}$

is a group with respect to permutation multiplication as composition.

Let $(ab) \in P_3$

Normalizer of (ab)

$\Rightarrow N(ab)$ = Set of all those elements of P_3 which commute with (ab).

$N(ab) = \{x \in P_3 \mid x(ab) = (ab)x\}$

Let $I \in P_3 \Rightarrow I(ab) = (ab)I \Rightarrow (ab) \in N(ab)$

$(bc)(ab) = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}\begin{pmatrix} a & c & b \\ b & c & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$

$= (a\,b\,c)$ —— (1)

and $(ab)(bc) = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} b & a & c \\ c & a & b \end{pmatrix} = (a\,c\,b)$ —— (2)

(1) & (2) $\Rightarrow (bc)(ab) \neq (ab)(bc) \Rightarrow (bc) \notin N(ab)$

Scanned by CamScanner

pdf-02 (08) Algebra, Dr Satish Kr Gupta

$(ca)(ab) = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}\begin{pmatrix} c & b & a \\ c & a & b \end{pmatrix} = \cancel{(a\ c)}\begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} = (a\ c\ b)$ — (3)

$(ab)(ca) = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} b & a & c \\ b & c & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} = (a\ b\ c)$ — (4)

(3) & (4) $\Rightarrow (ca)(ab) \neq (ab)(ca) \Rightarrow (ca) \notin N(ab)$

$(abc)(ab) = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} b & c & a \\ a & c & b \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} = (b\ c)$ — (5)

$(ab)(abc) = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} b & a & c \\ c & b & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} = (a\ c)$ — (6)

~~$(abc)(ab) = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} b & c & a \\ a & c & b \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} = (b\ c)$~~

(5) & (6) $\Rightarrow (abc)(ab) \neq (ab)(abc) \Rightarrow (abc) \notin N(ab)$

$(acb)(ab) = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}\begin{pmatrix} c & a & b \\ c & b & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} = (a\ c)$ — (7)

$(ab)(acb) = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} b & a & c \\ a & c & b \end{pmatrix} = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} = (b\ c)$ — (8)

(7) & (8) $\Rightarrow (acb) \notin N(ab)$.

$\therefore N(ab) = \{I, (ab)\}$

To show $N(ab)$ is not a normal subgroup of P_3

Let $(bc) \in P_3$, $(ab) \in N(ab)$

$(bc)(ab)(bc)^{-1} = (bc)(ab)(cb) = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$

$= \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}\begin{pmatrix} a & c & b \\ b & c & a \end{pmatrix}\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$

$= \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} b & c & a \\ c & b & a \end{pmatrix} = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} = (a\ c) \notin N(ab)$.

(9) Let N_1 and N_2 be two normal subgroups of a group G. Prove that $\frac{G}{N_1} = \frac{G}{N_2}$ iff $N_1 = N_2$.

Solution. "~~if part~~" Let N_1 and N_2 be any two normal subgroups of a group G

"if part" if $N_1 = N_2$

then obvious $N_1 a = N_2 a \Rightarrow \frac{G}{N_1} = \frac{G}{N_2}$ [$\because N_1 = N_2$]

only if part: Suppose $\frac{G}{N_1} = \frac{G}{N_2}$, To prove $N_1 = N_2$

$\frac{G}{N_1} = \{N_1 a \mid a \in G\}$ — (1)

$\frac{G}{N_2} = \{N_2 b \mid b \in G\}$ — (2)

~~if $N_1 = N_1$~~ put $a = b = e$, we have

$N_1 e = N_2 e \Rightarrow N_1 = N_2$

as e is common to both ($\frac{G}{N_1}$ and in $\frac{G}{N_2}$), proved.

Scanned by CamScanner

(16) Let Z denote the centre of a group G. If $\frac{G}{Z}$ is cyclic Then prove that G is abelian

Sol. Let Z be The centre of a group G

Let $\frac{G}{Z} = \{Za\}$ be a cyclic group

To prove G is abelian

Let $(Za)^n \in \frac{G}{Z}$ for some integer n

$\Rightarrow Za^n$ [$\because Z \trianglelefteq G$, $(Za)(Zb) = Zab$ etc

$(Za)(Za)(Za)\cdots(Za) = Zaa\cdots a = Za^n$ (n terms)

Let $e \in Z$ [$Za = \{\cancel{z \in Z, a \in G}\}$ $= \{za \mid z \in Z\}$]

then $Za = ea = Z$

$a \in Za \Rightarrow ea \in Za$ [$e \in Z$]

$\Rightarrow a = z_1 g^m$ for some $z_1 \in Z$

Similarly $b = z_2 g^n$ for some $z_2 \in Z$

$ab = z_1 g^m z_2 g^n = z_1 z_2 g^m g^n$ [Z is centre]

$ab = \cancel{z_1 g^m g^n}$

$ab = z_2 z_1 g^m g^n$

$ab = z_2 z_1 g^n g^m$

$ab = z_2 g^n z_1 g^m$

$ab = ba$

(13) If p is prime and G is non abelian of order p^3. Show that the centre of G has exactly p elements.

Soln. Let G be non abelian group such that

$O(G) = p^3$ —— (1)

Let Z be The centre of G

then $O(Z) > 1$ [$\because Z \neq \{e\}$]

By Lagrange's

$O(Z)$ must divide $O(G)$ [$\because Z \trianglelefteq G$]

Three cases arise (1) $O(Z) = p^3$ (2) $O(Z) = p^2$ (3) $O(Z) = p$

Scanned by CamScanner

(1) If $o(z) = p^3 \Rightarrow z = G \Rightarrow G$ is abelian, which is a contradiction

(2) If $o(z) = p^2$
then $o\left(\frac{G}{z}\right) = \frac{o(G)}{o(z)} = \frac{p^3}{p^2} = p$
$\Rightarrow \frac{G}{z}$ is prime order $\Rightarrow \frac{G}{z}$ is cyclic
$\Rightarrow G$ is abelian, which is again contradiction

Hence only possibility
is $o(z) = p$.
$\Rightarrow z$ has exactly p elements.

Q. (1) Define homomorphism (2) Define endomorphisms

$(G, *) \rightarrow$ group, $(G', o) \rightarrow$ group

A mapping $f : (G, *) \longrightarrow (G', o)$
is called homomorphism
if $f(a * b) = f(a) \, o \, f(b)$, $a \in G$, group, $f(a) \in G'$, group

A mapping $f : (G, *) \longrightarrow \cancel{(G', *)}\ (G, *)$ [$*$ or o]
is called endomorphism if
$f(a * b) = f(a) * f(b)$, $\forall a \in G$, also $f(a) \in G$

Note. (1) A homomorphism mapping which is one-one is called an isomorphism mapping

(2) Some authors say A homomorphism mapping is called isomorphism if this is one-one, onto.

Q. Define Kernel of homomorphism

let $\cancel{(G,*)}$ G and G' be any two groups
Then if K be the Kernel of homomorphism f
then $K = \{ x \in$ domain of f such that $f(x) = e'$, e' is identity in $G' \}$

Note. Since $f(e) = e' \Rightarrow e \in K$, i.e. $K \neq 0$

K is a subgroup of G
Let $a \in K \Rightarrow f(a) = e'$, $b \in K \Rightarrow f(b) = e'$
$\Rightarrow f(b^{-1}) = (f(b))^{-1} = \cancel{(e')^{-1} = e'}$
$= (e')^{-1} = e'$
$f(ab^{-1}) = f(a) f(b^{-1}) = e' . e' = e'$
$\Rightarrow ab^{-1} \in K \Rightarrow K$ is a subgroup of G.

To prove $K \trianglelefteq G$

Let $x \in G$, $k \in K \Rightarrow f(k) = e'$

To Show $x k x^{-1} \in K$

$f(x k x^{-1}) = f(x) f(k) f(x^{-1})$, f is homomorphism $[f(k) = e']$

$= f(x) e' . [f(x)]^{-1}$ $\quad [f(x^{-1}) = [f(x)]^{-1}]$

$= f(x) [f(x)]^{-1}$ $\quad [e'. f(x) = f(x) = f(x) . e']$

$= e'$

$\Rightarrow x k x^{-1} \in K$

$\Rightarrow K \trianglelefteq G$.

Theorem 03. The necessary and sufficient condition for a homomorphism f of a group G into a G' with Kernel K to be an isomorphism of G into G' is $K = \{e\}$.

Proof. Let $f: G \longrightarrow G'$ be a homomorphism where G and G' be any two groups.

Let $K = \{x \in G \mid f(x) = e', e' \in G'$, identity in $G'\}$

Necessary condition. Suppose $f: G \longrightarrow G$ be an isomorphism to prove $K = \{e\}$

Let $a \in K \Rightarrow f(a) = e'$

$\Rightarrow f(a) = f(e)$ $\quad [\because e' = f(e)$, e is identity in G

$\Rightarrow a = e$, $\because f$ is homomorphism$]$

~~Suff~~ Sufficient condition.

Let $a = e$

To prove f is an isomorphism

i.e to prove f is one-one

$f(a) = f(b) \Rightarrow f(a) [f(b)]^{-1} = f(b) [f(b)]^{-1}$

$\Rightarrow f(a) f(b^{-1}) = e'$ $\quad [f(b) [f(b)]^{-1} = e']$

$\Rightarrow f(a b^{-1}) = e'$ $\quad [f$ is homomorphism$]$

$\Rightarrow a b^{-1} \in K$

$\Rightarrow a b^{-1} = e$ $\quad [\because$ ...

$\Rightarrow a b^{-1} b = e b$

$\Rightarrow a e = e b$

$\Rightarrow a = b \Rightarrow f$ is one-one

Hence f is an isomorphism proved.

Theorem. Suppose N is a normal subgroup of G group G.

Let $f: G \longrightarrow \frac{G}{N}$ such that

$$f(x) = Nx, \quad \forall x \in G$$

then f is homomorphism and Kernel of $f = N$

Proof. Let N be a normal subgroup of a group G

Let $f: G \longrightarrow \frac{G}{N}$ such that

$$f(x) = Nx, \quad \forall x \in G$$

To prove f is homomorphism

$$f(xy) = \cancel{Nx\,Ny}\ Nxy$$

$$= Nx\,Ny \quad \left[\because N \trianglelefteq G \Rightarrow Nx\,Ny = Nxy\right]$$

$$f(xy) = f(x)f(y)$$

$\Rightarrow f$ is homomorphism.

Let $K = \{x \in G \mid f(x) = \text{Identity element of } \frac{G}{N} = N\}$

To prove $K = N$

Let $k \in K \Rightarrow f(k) = N$

$$\Rightarrow Nk = N \quad [\because f(k) = Nk, \text{ by def of } f]$$

$$\Rightarrow k \in N$$

$\therefore k \in K \Rightarrow k \in N \Rightarrow K \subseteq N$ —— (1)

Again Let $n \in N \Rightarrow f(n) = Nn$

$\Rightarrow$ But if $n \in N \Rightarrow \cancel{Nn = N}$ $Nn = N$

$\therefore \Rightarrow f(n) = N \quad [\because Nn = N]$

$n \in N \Rightarrow n \in K$

$\Rightarrow N \subseteq K$ —— (2)

(1) & (2) $\Rightarrow$ $\underline{N = K}$.

Scanned by CamScanner